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ON THE RELATIVE APPROACH OF TWO-DIMENSIONAL ELASTIC BODIES IN CONTACT

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Abstract—The classical method of Hertz fails to approximate the relative approach of bodies in a state of plane strain or plane stress induced by their mutual contact. The method also fails to provide a uniform approximation to the displacement field over an entire body. In this paper, the method of Hertz is interpreted in terms of modern perturbation theory. The relative approach is found for frictionless contact between a pair of elastic, circular cylinders which have differing radii and properties. A composite asymptotic expansion of the displacement field is developed, which is uniformly valid over both the neighborhood of the contact surface and the remainder of a body. The method employed in this paper may be applied to a wide variety of contact problems.

1. INTRODUCTION

THE relative approach [1] of a pair of three-dimensional elastic bodies in contact across a small smooth surface may be approximated to first order by the method of Hertz [2]. In two dimensions the method is inadequate because it yields the displacement field of the contact surface only to within an unknown constant [3]. It is the purpose of this paper to remedy this inadequacy and to interpret the method of Hertz in terms of modern perturbation theory.

Attention may be confined to either of the bodies in contact. The characteristic overall length, L, of the body is the primary reference length of the contact problem. The characteristic length, ℓ, of the contact surface is a secondary reference length. The distance, r^*, from a point of the contact surface to a field point, may be referred to either reference length by forming either of the dimensionless distances, r^*/L, or r^*/ℓ. An asymptotic expansion of the exact displacement field, in the small parameter ℓ/L, may be formed which is valid in the limit as ℓ/L vanishes, with r^*/L and the resultant contact force fixed. It is the physical meaning of this limit that the distribution of contact traction becomes concentrated at a point where the terms of the expansion are singular. The expansion is called an "outer" expansion because it is expected to approximate the exact displacement field only in an "outer" region; i.e. a region outside some neighborhood of the contact surface.

An asymptotic expansion in ℓ/L of the exact displacement field may also be formed which is valid in the limit as ℓ/L vanishes, but now with r^*/ℓ fixed. It is the physical meaning of this limit that contact occurs over a finite area, with the overall body becoming a semi-infinite region. The expansion is expected to approximate the exact solution in some neighborhood of the contact surface, but not necessarily over the whole body. This expansion is called an "inner" expansion; the region of its validity, the "inner" region.

It is the method of Hertz to formulate the *first order* problem of each expansion directly, through physical reasoning, without recourse to a formal perturbation scheme. The first order "outer" problem is that of the original body under a concentrated force equal to the resultant of the original distribution of contact traction. The first order "inner" problem is that of a half-space in contact over a finite area of its surface. Thus, the original contact problem is broken into a pair of problems, neither of which possesses, in general, a solution uniformly valid over the entire body originally considered. The presumption, of course, is that the solutions to these two problems may be recombined to form a uniform approximation to the displacement field. In particular, the relative approach for the original contact problem is sought. This quantity relates the resultant contact force to the displacement of the contact surface relative to a point of the outer region. To determine the relative approach, the displacement field for the half-space must be related to the displacement field for the body under concentrated force.

In three dimensions physical reasoning suffices, at least to first order. The deformation in the outer region of the original body is negligible compared to that in the inner region. For the purpose of finding the relative approach, the deformation for the body under concentrated force is neglected. In the half-space problem the translation at infinity is a readily identifiable constant because the remainder of the displacement field dies out as $1/r$, as the distance r from a point of the contact surface becomes infinite. By associating the point at infinity of the half-space with any point of the outer region, the displacement field of the half-space is uniquely determined. The relative approach follows.

The procedure sketched above may be shown to coincide with the first step in a formal perturbation procedure. Moreover, unless higher approximations are sought, such a formalism is not required in three dimensions. However, in the case of two dimensions, Hertz' method fails to provide the required asymptotic expansion, even to first order. The purpose of this paper is to formalize Hertz' method in terms of singular perturbation theory and thereby provide a prescription for the relative approach in two dimensions.

In two dimensions physical reasoning fails because the displacement field of the half-plane, rather than dying out at infinity, grows as $\log r$. This behavior in the leading term of the inner expansion prevents an intuitive recombination of the half-plane or inner solution with the outer solution under concentrated force. As a result, the relative approach in two dimensions cannot be approximated satisfactorily by the classical method of Hertz. However, the relative approach in two dimensions, as well as an asymptotic expansion for the displacement field which is uniformly valid over the entire domain of interest, may be found by straightforward application of singular perturbation theory [4]. In particular, the method of matched asymptotic expansions will be used to relate the inner and outer expansions, thereby determining the relative displacement of the contact surface with respect to a point in the outer region. A rigorous discussion of the matching of asymptotic expansions has been presented by Fraenkel [5] and will not be discussed here. The emphasis in this paper will be on the application of the theory to the problem at hand.

In the second subdivision of this paper the method of matched asymptotic expansions is applied to the particular problem of an elastic circle indented without friction, along antipodal arcs, by a pair of rigid circles. The relative approach between either rigid circle and the center of the elastic one is found, to first order, by matching the displacement field for the latter, subject to a pair of diametrically opposed concentrated forces, to the displacement field of a half-plane indented by a convex, rigid stamp. The relative approach of a pair of elastic circles, of differing radii and elastic properties, is found by simple adaptation of the previous result. A composite displacement field, uniformly valid over the entire domain of interest, is constructed from the displacement fields of the half-plane and the circle under concentrated force. Comparison of the inner displacement field with the composite reveals that the inner is not uniformly valid to first order. The outer expansion is singular; *a fortiori*, it is non-uniformly valid. The inner, outer and composite displacement fields are compared graphically.

In the third subdivision of this paper the displacement field is presented for a circle, in equilibrium under two opposed distributions of unidirectional traction, over antipodal boundary arcs. An inner expansion is formed which, to first order, is the displacement field of the half-plane, subject to distributed traction. If the traction distribution appropriate to the half-plane indentation problem is employed, it may be shown that the inner displacement field of the second subdivision of this paper is recovered, even to the rigid body translation determined by matching. In addition, an outer expansion is formed which, to first order, is the displacement field of the circle, subject to antipodal concentrated forces. This is so, provided that the applied traction satisfies a condition met by a broad range of contact problems; viz. that each distribution of applied traction be of one sense over its entire arc. For a more general distribution of traction, the outer expansion to first order could be the displacement field for the circle, subject to a combination of concentrated moments of various orders. Thus, a simple-minded attempt to apply Saint Venant's principle, by replacing distributed traction with a statically equivalent force and couple combination alone, could result in an incorrect first order outer expansion. However, it is expected that such error would reveal itself in the matching process.

The results of this paper may be generalized to other contact problems. The body of existing solutions to half-plane contact problems may be matched to available solutions for bodies of finite extent, subject to concentrated forces.

2. THE CIRCLE IN CONTACT

Let (x^*, y^*) be a set of rectangular Cartesian coordinates, with the complex variable z^* defined by

$$z^* \equiv x^* + iy^*. \tag{2.1}$$

An isotropic, elastic, circular cylinder of infinite length and radius R, and a thin, isotropic, elastic disc of the same radius are considered alternatively. The right section of either body has its center at $z^* = iR$ (Fig. 1).

Two rigid circles with centers on the imaginary axis, having radii R' and R'', indent the elastic circle from below and above, respectively. Only the lower arc of contact will be considered explicitly. For simplicity, it is assumed that indentation takes place without friction. Each rigid circle exerts on the elastic one a resultant contact force F^* per unit depth of the plane figure, thereby inducing in the long cylinder a state of plane strain; in the thin disc a state of generalized plane stress is induced. Let u^* and v^* be the components of displacement in the x^* and y^* directions, respectively, and define

$$w^* = u^* + iv^*. \tag{2.2}$$

Let λ^* denote the relative approach of the rigid circle of radius R' to the center of the elastic circle, in the process of indentation.

The elastic circle is in contact with the lower rigid circle along a length ℓ, which is as yet an unknown function of F^*. The validity of the method of Hertz, as compared to an

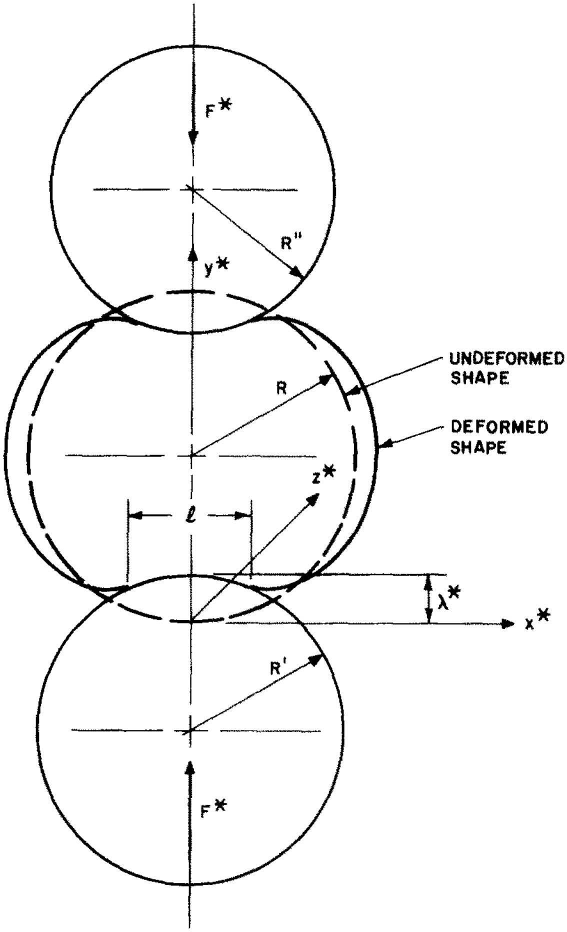

FIG. 1. Elastic circle indented by rigid circles.

exact solution of the equations of linear elasticity, requires that ℓ be small compared to the radius R of the elastic circle. This assumption gives rise to the small perturbation parameter ε, defined as

$$\varepsilon \equiv \ell/R. \tag{2.3}$$

The applicability of the equations of linear elasticity themselves requires that ℓ be small compared to the sum of the radii R and R'. Thus, the small auxiliary parameter

$$\varepsilon' \equiv \ell/R' = O(\varepsilon) \tag{2.4}$$

is required. The equation of the undeformed boundary of the lower half of the circle is given by

$$y^*(x^*) = R - \sqrt{(R^2 - x^{*2})}. \tag{2.5}$$

To an accuracy of $O(\varepsilon)$ in the interval $|x^*| \leq \ell/2$,

$$y^*(x^*) \sim x^{*2}/2R, \qquad |x^*| \leq \ell/2. \tag{2.6}$$

Similarly, the equation of the upper surface of the lower rigid circle, when a single point of contact exists, is given by

$$y^*(x^*) \sim -x^{*2}/2R', \qquad |x^*| \le \ell/2. \tag{2.7}$$

A most important consequence of the approximations (2.6) and (2.7) is that only the curvature of each circle at the initial point of contact appears. No other feature of geometry enters. As a result, the inner displacement field to $O(1)$, as found by the method of Hertz, will be broadly applicable to a very wide variety of overall body shapes.

In the interval $|x^*| \le \ell/2$, the arc given by (2.6) is deformed into the shape (2.7). Therefore, the displacement of the contact arc in the direction of the positive imaginary axis is given to $O(\varepsilon)$ by

$$v^*(x^*) = -\frac{1}{2}\left(\frac{1}{R}+\frac{1}{R'}\right)x^{*2}+\text{const.}, \qquad |x^*| \le \ell/2, \tag{2.8}$$

where the constant is unknown. Relation (2.8) is a boundary condition of the indentation problem. A like condition exists over the upper contact arc of the elastic circle. The remaining boundary conditions for the problem of frictionless indentation all require zero traction.

Inner variables are now formed which have the length ℓ of the contact arc as the reference length. Define

$$Z \equiv z^*/\ell, \qquad W \equiv w^*/\ell, \qquad \Lambda \equiv \lambda^*/\ell, \qquad F \equiv F^*/\mu\ell, \qquad P \equiv p^*/\mu, \tag{2.9}$$

where μ is the shear modulus of the elastic material and p^* is a surface traction, whose positive sense is into the elastic circle. The limit as the perturbation parameter ε vanishes, with Z fixed, maintains the length of the contact arc, ℓ, as the characteristic reference length. The corresponding asymptotic expansion, valid in the region surrounding the contact arc, is referred to as the inner expansion.

Corresponding outer variables are defined which have the radius R of the circle as the reference length:

$$z \equiv z^*/R, \qquad w \equiv w^*/R, \qquad \lambda \equiv \lambda^*/R, \qquad f \equiv F^*/\mu R, \qquad p \equiv p^*/\mu. \tag{2.10}$$

The limit as ε vanishes, with z fixed, maintains an overall characteristic length of the body as the reference length. The corresponding expansion is valid in the outer region. The inner and outer quantities are related by

$$z = \varepsilon Z, \qquad w = \varepsilon W, \qquad \lambda = \varepsilon\Lambda, \qquad f = \varepsilon F, \qquad p = P. \tag{2.11}$$

In accord with the method of Hertz, the inner region to $O(1)$ is taken to be the upper half of the Z-plane. On the scale of length of the inner region, the indentation (2.8) becomes

$$V_i(X) = -\tfrac{1}{2}(\varepsilon + \varepsilon')X^2 + \text{const.}, \qquad |X| \le \tfrac{1}{2}. \tag{2.12}$$

The relation (2.12) is the first-order indentation caused by a rigid, convex stamp with curvature $\varepsilon+\varepsilon'$ at its apex. The solution to this indentation problem, with the non-unique displacement field unspecified to an unknown constant, is given by [3, 6]

$$4\pi W_i(Z) = C - iF\{\kappa[2ZQ(Z)-\log Q(Z)]+2\bar{Z}Q(\bar{Z})-\log Q(\bar{Z})+4(Z-\bar{Z})Q(\bar{Z})\}, \tag{2.13}$$

where C is an unknown constant, κ is given by

$$\kappa = 3-4\nu, \qquad \text{(plane strain)}, \tag{2.14a}$$

$$\kappa = (3-\nu)/(1+\nu), \qquad \text{(plane stress)}, \tag{2.14b}$$

and ν is Poisson's ratio. The function Q is defined by

$$Q(Z) \equiv 2Z - i\sqrt{(1-4Z^2)}, \tag{2.15}$$

in which $\sqrt{(1-4Z^2)}$ is holomorphic in the Z-plane cut along the interval $[-\frac{1}{2}, \frac{1}{2}]$ of the real axis. The branch chosen is that which takes positive values as the cut is approached from above. The asymptotic behavior of this branch is given by

$$\sqrt{(1-4Z^2)} = -2iZ\left(1-\frac{1}{4Z^2}\right)^{\frac{1}{2}} \sim -2iZ\left(1-\frac{1}{8Z^2}\right), \qquad |Z| \to \infty. \tag{2.16}$$

Finally, throughout this paper the symbol "log" denotes the principal branch; i.e.

$$\log Q \equiv \log|Q| + i \arg Q, \qquad -\pi < \arg Q \leq \pi. \tag{2.17}$$

It may be verified that (2.13) satisfies the indentation condition (2.12) with the aid of the relation [6]

$$F = \pi(\varepsilon+\varepsilon')/2(\kappa+1). \tag{2.18}$$

[Relation (2.18) gives the force required to bring a stamp of curvature $\varepsilon+\varepsilon'$ into contact with the half-plane along a unit of length.] In the next subdivision, it will be shown that (2.13) is the inner expansion to $O(1)$, as ε vanishes with Z fixed.

In accord with the method of Hertz, it is assumed that the outer problem to $O(1)$ is that of the circle subject to a pair of concentrated compressive forces applied to the boundary at $z = 0$ and $z = 2i$. The solution to this problem has also been supplied by Hertz (see [6]). In outer variables the displacement field takes the form

$$\begin{aligned} 4\pi w_o(z) = & -if\{\kappa \log[2z/(2+iz)] + \log[2\bar{z}/(2-i\bar{z})] + z/\bar{z} + (2+iz)/(2-i\bar{z}) \\ & + (\kappa-1)(1+iz) - \kappa\} + C', \end{aligned} \tag{2.19}$$

where C' is a constant which may be chosen to make a particular point in the outer region; e.g. the center of the circle, the reference point for displacement. In the next subdivision it will be shown that (2.19) is the outer expansion of the displacement field to $O(1)$, as ε vanishes with z fixed.

The inner and outer expansions, detailed above, differ because they represent two different physical limits, both of which correspond to the vanishing of ε. In the inner limit the reference length ℓ is fixed as R becomes infinite, while in the outer, the reference length R is fixed as ℓ vanishes. Nonetheless, these two expansions share a region of common validity and may be matched to a given order in the perturbation parameter ε. The matching permits a determination of unknown constants in both expansions. In particular, the previously unknown rigid body translation C of the half-plane may be related to the translation C' associated with the outer region. The matching rule used here is that originally set forth by Van Dyke [4] and recently modified by Fraenkel [5]:

The inner expansion [to $O(\Delta)$] of the outer expansion [to $O(\delta)$] is identical to the outer expansion [to $O(\delta)$] of the inner expansion [to $O(\Delta)$]. Symbolically, this equality is written as

$$w_i]_o = w_o]_i.$$

The required expansions will now be formed.

In order to form $w_i]_o$, the inner variables of (2.13) are replaced by outer variables through the use of (2.11), and the result is expanded for small ε. For the purposes of this paper, the expansion to $O(1)$ is sufficient. Thus, the outer expansion to $O(1)$, of the inner expansion to $O(1)$, is given by

$$4\pi w_i]_o = \varepsilon C + if(\kappa+1)\log\varepsilon - if[\kappa\log z + \log\bar{z} + z/\bar{z} + 2(\kappa+1)\log 2 + \tfrac{1}{2}(\kappa-1)], \qquad \varepsilon\to 0. \tag{2.20}$$

The relation (2.20) may also be expressed in inner variables by means of (2.11). Thus,

$$4\pi W_i]_o = C - iF[\kappa\log Z + \log\bar{Z} + Z/\bar{Z} + 2(\kappa+1)\log 2 + \tfrac{1}{2}(\kappa-1)], \qquad |Z|\to\infty. \tag{2.20'}$$

The relation (2.20′) emphasizes that the outer expansion of the inner expansion to $O(1)$ gives the asymptotic behavior of the half-plane displacement field at infinity.

In order to form $w_o]_i$, the outer variables of (2.19) are replaced by inner variables and the result is again expanded for small ε. For the purposes of matching, the expansion to $O(1)$ is retained. Thus, the inner expansion to $O(1)$, of the outer expansion to $O(1)$, is given by

$$4\pi W_o]_i = C'/\varepsilon - iF[(\kappa+1)\log\varepsilon + \kappa\log Z + \log\bar{Z} + Z/\bar{Z}], \qquad \varepsilon\to 0. \tag{2.21}$$

The relation (2.21) may also be expressed in outer variables by means of (2.11). Thus,

$$4\pi w_o]_i = C' - if(\kappa\log z + \log\bar{z} + z/\bar{z}), \qquad |z|\to 0. \tag{2.21'}$$

The relation (2.21′) emphasizes that the inner expansion of the outer expansion to $O(1)$ gives the asymptotic behavior, near a singular point, of the displacement field for the circle under concentrated forces.

Equations (2.20) and (2.21′) may be matched, with the result that

$$\varepsilon C - C' = if[(\kappa+1)(2\log 2 - \log\varepsilon) + \tfrac{1}{2}(\kappa-1)]. \tag{2.22}$$

The relative approach of the rigid circle to the center of the elastic circle is equal to the imaginary part of the displacement of the half-plane at $Z = 0$ with respect to the center of the elastic circle. These two displacements are found from (2.13) and (2.19) respectively, and (2.22) is then used to eliminate the quantity $\varepsilon C - C'$. The relative approach is given by

$$4\pi\lambda = -\frac{\kappa+1}{2}f\log(e\varepsilon^2/64), \tag{2.23}$$

where e is the base of natural logarithms. The length ℓ of the contact arc, which appears in ε, may be eliminated from (2.23) by use of (2.18), (2.11), (2.4) and (2.3). The relative approach of the rigid circle to the center of the elastic one is given in convenient dimensionless form by

$$\tilde{\lambda} = -\tilde{f}\log\tilde{f}, \tag{2.24}$$

where

$$\tilde{\lambda} \equiv e\lambda^*/4R(1+R/R'), \tag{2.25a}$$

$$\tilde{f} \equiv e(\kappa+1)F^*/32\pi\mu R(1+R/R'). \tag{2.25b}$$

The relative approach of a rigid circle to the center of the elastic one, given by (2.24) and (2.25), may be adapted readily to the case of contact between a pair of elastic circles of differing radii and properties. Let R' be the radius of curvature of the common contact arc in the deformed state. Effectively, the first elastic circle behaves as if indented by a rigid circle of radius R'; the second elastic circle, by a rigid circle of radius $(-R')$. R' may be found by writing the contact force–indentation relation, (2.18), in dimensional form for both elastic circles, and then equating the resultant contact forces. Then R' may be eliminated from (2.25) as applied to the first elastic circle and as applied to the second. The relative approach between the centers of the two elastic circles is then given by

$$\hat{\lambda} = -\frac{\eta_1}{\eta_1+\eta_2}\hat{f}\log\left(\frac{R_2}{R_1}\hat{f}\right) - \frac{\eta_2}{\eta_1+\eta_2}\hat{f}\log\left(\frac{R_1}{R_2}\hat{f}\right), \tag{2.26}$$

where

$$\hat{\lambda} = e(\lambda_1^* + \lambda_2^*)/4(R_1+R_2), \tag{2.27a}$$

$$\hat{f} = eF^*(\eta_1+\eta_2)/32\pi(R_1+R_2), \tag{2.27b}$$

and

$$\eta_i \equiv (\kappa_i+1)/\mu_i, \qquad i = 1, 2. \tag{2.28}$$

Expressions for relative approach having been found, it remains to construct an expansion for the displacement field which is uniformly valid over both the inner and outer regions which comprise the lower half of the indented elastic circle.

Neither the inner nor the outer expansion is necessarily uniformly valid over all of the lower half of the elastic circle; certainly, the logarithmic singularity of the outer expansion, at a point of concentrated force, insures its non-uniform validity. This defect, as well as the non-uniform approximation offered by the inner expansion, (2.13), is remedied by a composite expansion, constructed from the inner and outer according to the rule [4, 5]

$$w_c = w_i + w_o - w_i]_o.$$

Substitution of (2.13), written in outer variables by means of (2.11); (2.19), where C' is adjusted to give zero displacement at the center of the circle; and (2.20), yields the composite expansion:

$$\begin{aligned} 4\pi w_c = -if\Bigg\{\kappa\left[2\frac{z}{\varepsilon}Q\left(\frac{z}{\varepsilon}\right) - \log\left(\frac{1}{\varepsilon}Q\left(\frac{z}{\varepsilon}\right)\right) - \log(2+iz)\right] + 2\frac{\bar{z}}{\varepsilon}Q\left(\frac{\bar{z}}{\varepsilon}\right) - \log\left(\frac{1}{\varepsilon}Q\left(\frac{\bar{z}}{\varepsilon}\right)\right) - \log(2-i\bar{z}) \\ +4\left(\frac{z-\bar{z}}{\varepsilon}\right)Q\left(\frac{\bar{z}}{\varepsilon}\right) + \frac{2+iz}{2-i\bar{z}} + i(\kappa-1)z + \frac{1}{2}(\kappa-1)(1-i\pi) - 2(\kappa+1)\log 2\Bigg\}, \end{aligned} \tag{2.29}$$

in which f and ε are related by the contact force–indentation relation (2.18). At the center of the circle, $z = i$, w_c differs from zero by $O(\varepsilon^2)$, while at the center of the contact arc, $z = 0$, it gives the relative approach, (2.24). The inner expansion w_i differs from the composite w_c by a function whose value is zero at $z = 0$, but whose least upper bound is of $O(1)$ in the region of interest. Therefore, w_i is not a uniform approximation to $O(1)$.

Figure 2 compares the composite expansion, (2.29), the outer expansion, (2.19), and the inner expansion, (2.13), written in outer variables. For each, the y-component of displacement, v, has been calculated along the y-axis, from the contact point, $y = 0$, to the center of

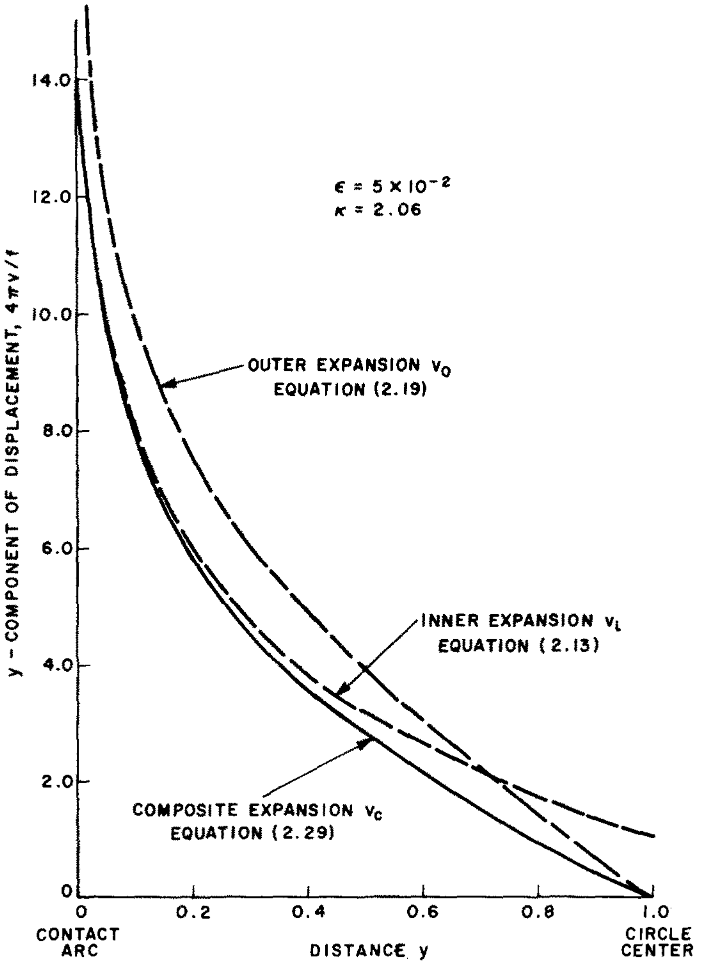

FIG. 2. Comparison of inner, outer and composite expansions.

the circle, $y = 1$. The value chosen for ε, 5×10^{-2}, is very nearly the value at which plastic yielding would be initiated in a cylinder of a chromium–vanadium steel [7], in a state of plane strain, compressed between a pair of parallel, rigid planes ($\varepsilon' = 0$). The value, $\kappa = 2{\cdot}06$, is also a value typical of steel.

The inner expansion accumulates an error of $O(1)$ in the passage to the center of the circle, $y = 1$. The outer expansion is logarithmically singular at $y = 0$.

3. THE CIRCLE SUBJECT TO DISTRIBUTED TRACTION

As in the second subdivision of this paper, the right section of a cylindrical elastic body occupies the interior of a circle of radius R, lying in the z^*-plane, with its center at $z^* = iR$ (Fig. 3). In this subdivision, the circle is subject to a pair of opposed, but otherwise identical tractions, p^*, distributed over the respective intervals, $|x^*| \leq \ell/2$, of the upper and lower

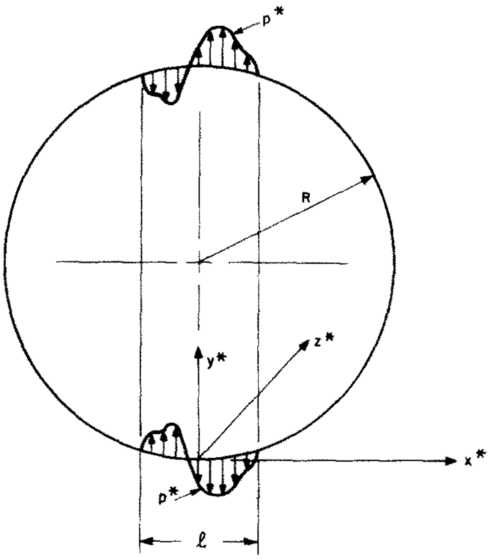

FIG. 3. Elastic circle subject to distributed traction.

boundary arcs, $y^* = R \pm \sqrt{(R^2 - x^{*2})}$. The distributions of traction p^* are parallel to the y^*-axis. Their positive sense is toward the interior of the circle.

The displacement field for the circle, subject to given traction, has been found by Hertz (see [6]). In the present case, the displacement field is given by

$$\mu w^*(z^*) = \int_{-\ell/2}^{\ell/2} p^*(s^*) G\left(\frac{s^*}{R}, \frac{z^*}{R}\right) ds^*, \tag{3.1}$$

where s^* is the source point counterpart of the field point variable x^*, and the Green's function G is given by

$$4\pi i G(s,z) = \kappa \log \frac{i(1-s^2)^{\frac{1}{2}} - i - (s-z)}{(1-s^2)^{\frac{1}{2}} + 1 - i(s-z)} + \log \frac{-i(1-s^2)^{\frac{1}{2}} + i - (s-\bar{z})}{(1-s^2)^{\frac{1}{2}} + 1 + i(s-\bar{z})} + \frac{i(1-s^2)^{\frac{1}{2}} - i - (s-z)}{-i(1-s^2)^{\frac{1}{2}} + i - (s-\bar{z})}$$
$$+ \frac{(1-s^2)^{\frac{1}{2}} + 1 - i(s-z)}{(1-s^2)^{\frac{1}{2}} + 1 + i(s-\bar{z})} + (\kappa - 1)(1-s^2)^{\frac{1}{2}}(1+iz) - (\kappa-1)i\pi/2. \tag{3.2}$$

The imaginary part of G vanishes at the center of the circle; the center is a reference point for the vertical component of displacement, v^*, as was true *a fortiori* in the preceding part of this paper.

The displacement field, (3.1), may be written in either inner or outer variables. In the respective expressions, the parameter ε will appear differently. Consequently, expansion of each expression for small ε will result in two different asymptotic representations. With the aid of (2.9), the displacement field is written in inner variables as

$$W(Z;\varepsilon) = \int_{-1/2}^{1/2} P(S) G(\varepsilon S, \varepsilon Z) \, dS \tag{3.3}$$

(see Fig. 4). Similarly, (2.10) is used to write (3.1) in outer variables (see Fig. 5). Thus,

$$w(z\,;\varepsilon) = \int_{-\varepsilon/2}^{\varepsilon/2} p(s\,;\varepsilon)G(s, z)\,\mathrm{d}s. \tag{3.4}$$

The Green's function of the displacement field in inner variables, (3.3), may be expanded asymptotically for small ε, in the form

$$G(\varepsilon S, \varepsilon Z) = \sum_{n=0}^{\infty} \Delta_n(\varepsilon)G_n(S, Z), \tag{3.5}$$

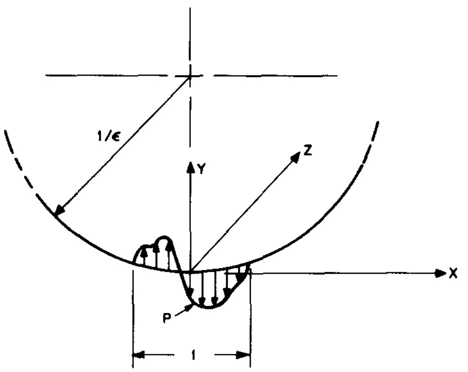

FIG. 4. Elastic circle in inner variables.

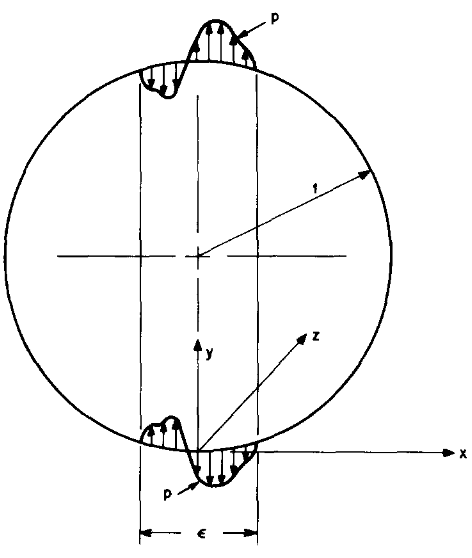

FIG. 5. Elastic circle in outer variables.

where the G_n are given by

$$G_0(S, Z) = \lim_{\varepsilon\to 0} G(\varepsilon S, \varepsilon Z)/\Delta_0(\varepsilon),$$
$$G_n(S, Z) = \lim_{\varepsilon\to 0} \frac{G(\varepsilon S, \varepsilon Z) - \sum_{m=0}^{n-1} \Delta_m(\varepsilon) G_m(S, Z)}{\Delta_n(\varepsilon)}, \qquad n > 0, \tag{3.6}$$

and the "gauge" functions Δ_n are taken to be

$$\Delta_n(\varepsilon) \equiv \begin{cases} \log \varepsilon, & n = 0, \\ \varepsilon^{n-1}, & n > 0. \end{cases} \tag{3.7}$$

Thus, the formal inner expansion of the displacement field is given by

$$4\pi W(Z;\varepsilon) = -iF(\kappa+1)\log\varepsilon + iF[(\kappa+1)\log 2 - \kappa + (\kappa-1)i\pi/2]$$
$$-i\int_{-1/2}^{1/2} P(S)\left[\kappa\log(Z-S) + \log(\bar{Z}-S) + \frac{Z-S}{\bar{Z}-S}\right] dS + O(\varepsilon). \tag{3.8}$$

The bracketed expression, which appears in the integral of (3.8), is the Green's function appropriate to the upper half-plane subject to a distribution of normal traction over the boundary [6]. The appearance of the half-plane in the first approximation is elucidated by inspection of Fig. 4. It may be shown that the displacement field, (2.13), of the half-plane indentation problem is recovered upon substitution of the appropriate traction distribution [3, 6],

$$P \equiv \frac{4}{\pi} F\sqrt{(1-4S^2)}, \tag{3.9}$$

into (3.8), truncated at $O(1)$. Recovery is complete, even to a rigid body translation; the displacement at $Z = 0$ is in exact agreement with the relative approach, (2.23).

The displacement field in outer variables, (3.4), may also be formally expanded for small ε, in the form

$$w(z;\varepsilon) = \sum_{n=1}^{\infty} \delta_n(\varepsilon)\phi_n(z), \tag{3.10}$$

where

$$\phi_n(z) = \lim_{\varepsilon\to 0} \frac{w(z;\varepsilon) - \sum_{m=1}^{n-1} \delta_m(\varepsilon)\phi_m(z)}{\delta_n(\varepsilon)}, \tag{3.11}$$

and the gauge functions δ_n are taken to be

$$\delta_n(\varepsilon) \equiv \varepsilon^{n-1}, \qquad n > 0. \tag{3.12}$$

In the limiting process of (3.11), it is required that the resultant force, f, of each distribution of traction, p, be held constant while the arc over which p is applied shrinks to a point. Thus,

$$\lim_{\varepsilon\to 0} \int_{-\varepsilon/2}^{\varepsilon/2} p(s;\varepsilon)\, ds = f. \tag{3.13}$$

If the function p also satisfies an additional condition [8, 9]; e.g.

$$p(s;\varepsilon) \geq 0, \tag{3.14}$$

then, in the mathematical sense of a distribution [8], the distribution p will converge to

$$\lim_{\varepsilon \to 0} p = f\delta, \tag{3.15}$$

where δ is the Dirac distribution. In the contact, or indentation problem of this paper, (3.14), and therefore (3.15), is satisfied because the contact traction must be compressive over the entire contact arc.

Now, according to (3.11) and (3.4), the outer expansion to $O(1)$ is given by

$$4\pi\phi_1(z) = 4\pi \lim_{\varepsilon \to 0} w(z, \varepsilon) = \lim_{\varepsilon \to 0} \int_{-\varepsilon/2}^{\varepsilon/2} p(s;\varepsilon)G(s, z)\,\mathrm{d}s. \tag{3.16}$$

It follows from the definition of the symbolism of (3.15), that

$$4\pi\phi_1(z) = \lim_{\varepsilon \to 0} \int_{-\varepsilon/2}^{\varepsilon/2} p(s;\varepsilon)G(s, z)\,\mathrm{d}s = fG(0, z). \tag{3.17}$$

Reference to (3.2) indicates that the displacement field of the circle subject to concentrated forces, (2.19), has been recovered, as suggested by inspection of Fig. 5, under the restriction $p \geq 0$.

It should be noted that, in general, the distribution, p, gives rise to moments m_n of all orders, defined as

$$m_n(\varepsilon) \equiv \int_{-\varepsilon/2}^{\varepsilon/2} s^n p(s;\varepsilon)\,\mathrm{d}s, \qquad n = 0, 1, 2, \ldots. \tag{3.18}$$

For a distribution p, more general than that of the contact problem, any of the m_n may be of $O(1)$, as ε vanishes. In this case, the outer expansion of the displacement field, to $O(1)$, is a linear combination of nuclei of strain, $(\partial^n/\partial s^n)G(s = 0, z)$, each multiplied by its appropriate moment, $\lim_{\varepsilon \to 0} m_n(\varepsilon)$. Thus, an outer expansion valid to $O(1)$ is not necessarily obtained by replacing the distribution, p, with a statically equivalent system of concentrated force and couple, as might be done in an attempt to apply Saint Venant's principle. However, it is expected that any error arising in this way would reveal itself through a failure to match the inner expansion.

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Абстракт—Классический метод Герца не удобен для аппроксимации релятивного подхода для тел, в состоянии плоского напряжения или плоской деформации, вызванных их взаимным контактом. Он также не удобен для примения однородного приближения для поля перемещений по всем теле.

В пердлагаемой работе, представляется метод Герйа в форме интериретации современной теории возмущений. Дается релятивный подход для контакта, без трения между парой упругих, круглых цилиндров, с разными радиусами и свойствами. Определяется сложное, асимптотическое разложение для поля перемещений, важное как для окрестности поверхности контакта как и для остальной части тела. Метод, предложенный в работе, можно использавать для широкого круга контактных задач.